

# Uniqueness of the solution of the Gaudin's equations, which describe a one-dimensional system of point bosons with zero boundary conditions

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## Abstract

We show that the system of Gaudin's equations for quasimomenta  $k_j$ , which describes a one-dimensional system of spinless point bosons with zero boundary conditions, has the unique real solution for each set of quantum numbers  $n_j$ .

## 1 Introduction

The one-dimensional (1D) system of spinless point bosons is one of the most investigated integrable systems [1, 2, 3, 4, 5, 6, 7, 8, 9] (we give the basic references or those related directly to our work). This is the simplest system. Therefore, the understanding of its properties is of importance. The system is described by a wave function in the form of the Bethe ansatz with quasimomenta  $k_j$  [2, 5]. For the periodic boundary conditions (BCs), it takes the form [2]

$$\psi_{\{k\}}(x_1, \dots, x_N) = \sum_P a(P) e^{i \sum_{l=1}^N k_{P_l} x_l}, \quad (1)$$

where  $P$  means all permutations of  $k_l$ . The Schrödinger equation and BCs lead to the equations for  $k_j$ , which contain all information about the system: the ground-state energy, quasiparticle dispersion law, thermodynamic quantities, *etc.* For the proper determination of these quantities, it is important to know whether the solution  $\{k_j\}$  is unique for each collection of quantum numbers  $n_j$ . For the periodic boundary conditions (BCs), the uniqueness of the solution was proved by Yang and Yang [4] (more precisely, work [4] gave only the idea of a proof, and the strict proof was obtained by Takahashi [7]). However, the real systems have usually boundaries. *Zero* BCs are set for the domain  $0 \leq x_1 \leq x_2 \leq \dots \leq x_N \leq L$ , they read [5, 6]

$$\Psi(x_1 = 0, x_2, \dots, x_N) = \Psi(x_1, \dots, x_{N-1}, x_N = 0) = 0. \quad (2)$$

For such BCs, the solution should be sought in the form [5, 6]

$$\Psi_{\{|k|\}}(x_1, \dots, x_N) = \sum_{\{\varepsilon\}} C(\varepsilon_1, \dots, \varepsilon_N) \psi_{\{k\}}(x_1, \dots, x_N), \quad (3)$$

where  $\varepsilon_j = \pm 1$ . Gaudin showed [5, 6] that the wave function (3) satisfies BCs (2) and the Schrödinger equation, if the quasimomenta  $k_j$  satisfy the definite equations. We call them the Gaudin's equations and give them in the next section. On the basis of these equations, the ground state and excited levels of the system were studied, respectively, in [5, 8] and [9]. The Gaudin's equations possess a particular symmetry, which allows us to differently introduce quasiparticles and to construct the thermodynamics in a new way [9, 10]. But the uniqueness of the solution of the Gaudin's equations was not proved previously. This will be made in the present work.

## 2 Some properties of the Gaudin's equations

Consider the system of  $N$  spinless point bosons located on the segment of length  $L$ . The Schrödinger equation and BCs (2) yield for this system the following equations for quasimomenta  $k_j$  [5]:

$$e^{2ik_j L} = \prod_{l=1}^N \frac{(k_j + ic)^2 - k_l^2}{(k_j - ic)^2 - k_l^2} \Big|_{l \neq j}, \quad j = 1, \dots, N. \quad (4)$$

Here, the product contains no factor with  $l = j$ , all  $k_j$  are nonzero real numbers, and  $|k_l| \neq |k_j|$  for  $l \neq j$  (the case where  $k_j = 0$  for one or several  $j$  and the case where  $|k_l| = |k_j|$  for one or several pairs  $j, l$  are excluded, since the Schrödinger equation with zero BCs has no solutions in these cases). We consider only the repulsive interaction:  $c > 0$ . Equations (4) are invariant relative to the changes  $k_j \rightarrow -k_j$  and (independently)  $k_l \rightarrow -k_l$ . Therefore, it is sufficient to find the solutions with  $k_j > 0$  for all  $j$ . Equations (4) yield the Gaudin's equations [5]

$$Lk_i = \pi n_i + \sum_{j=1}^N \left( \arctan \frac{c}{k_i - k_j} + \arctan \frac{c}{k_i + k_j} \right) \Big|_{j \neq i}, \quad i = 1, \dots, N, \quad (5)$$

where  $n_i = 0, \pm 1, \pm 2, \dots$ , and  $k_i > 0$  for all  $i$ . If  $c = 0$ , these equations possess the solution for free bosons in a box:  $Lk_i = \pi n_i$ ,  $n_i = 1, 2, \dots$  ( $i = 1, \dots, N$ ). Based on the idea of the continuity of solutions in the interaction, Gaudin concluded [5] that Eqs. (5) should be solved only with positive values of  $n$ 's ( $n_i = 1, 2, \dots$ ), for which there exists a continuous transition  $c \rightarrow 0$  to the solutions for free bosons. We do not agree with this argument. Because at a fixed  $c > 0$  nothing prevents the system from passing from the state with  $n_i > 0$  (for all  $i$ ) in a state with  $n_i \leq 0$  (for one or several  $i$ ) due to the gradual (or jump-like) decrease of  $n$ 's. Therefore, the solutions corresponding to  $n_i \leq 0$  for one or several  $i$  should be taken into account. Let us clarify what are those solutions.

Let the set  $\{n_i\}$  has  $n_l < 0$  for a single  $l$ . We multiply the  $l$ th equation from (5) by  $-1$  and get

$$L\tilde{k}_l = \pi\tilde{n}_l + \sum_{j=1}^N \left( \arctan \frac{c}{\tilde{k}_l - k_j} + \arctan \frac{c}{\tilde{k}_l + k_j} \right) |_{j \neq l}, \quad (6)$$

where  $\tilde{n}_l = -n_l > 0$ ,  $\tilde{k}_l = -k_l$ . So, we return to an equation of the form (5), but already with a positive  $n_l$ . This means that the solution  $\{k_i\}$  for  $n_l = p < 0$  differs from the solution  $\{k_i\}$  for  $n_l = -p > 0$  only by the sign of  $k_l$ . However, the solutions, which differ from one another only by the sign of one or several  $k_i$ , are physically equivalent, because they lead to the same wave function  $\Psi(x_1, \dots, x_N)$  (3). Therefore, it is sufficient to find the solutions for all nonnegative  $n_i$ .

Consider the case, for which the set  $\{n_i\}$  contains a zero  $n_l$ . Suppose that  $k_l = 0$  in the  $l$ th equation in (5), then this equation is satisfied identically. We assign the number 1 to this equation. Then we have  $k_1 = 0$ , and the remaining equations take the form

$$Lk_i = \pi n_i + 2 \arctan \frac{c}{k_i} + \sum_{j=2}^N \left( \arctan \frac{c}{k_i - k_j} + \arctan \frac{c}{k_i + k_j} \right) |_{j \neq i}, \quad (7)$$

where  $i = 2, \dots, N$ . In the next section, we will show that the systems (5) and (7) have the unique solution. This implies that the solution with  $k_1 = 0$  and  $k_2, \dots, k_N$  from (7) coincides with the unique solution of system (5). If  $n_l = 0$  for several  $l$ , then the solution contains one zero  $k_i$ . However, it was noted above that all solutions  $\{k_i\}$  with  $k_l = 0$  for at least one  $l$  should be omitted.

Thus, the consideration of  $n_i \leq 0$  does not lead to physically new solutions. Therefore, in order to find all admissible nonequivalent solutions of system (4), it is sufficient to solve system (5) with  $n_i = 1, 2, \dots$  and  $k_i > 0$  for all  $i$ . This coincides with the conclusion made by Gaudin [5, 6].

### 3 Uniqueness of the solution of the Gaudin's equations

We will prove the uniqueness of the solution of the Gaudin's equations (5) with the help of the methods by Yang and Yang [4] and by Takahashi [7].

First, we recall the Gaudin's idea [5, 6] of that the equations for a periodic system [5]

$$Lk_i = 2\pi n_i + 2 \sum_{j=1}^N \arctan \frac{c}{k_i - k_j} |_{j \neq i}, \quad i = 1, \dots, N, \quad (8)$$

can be written in the form (5). Let  $N$  be even, and let the set of quantum numbers  $n_i$  is specularly antisymmetric:  $n_l = -n_{N-l+1}$ ,  $l = 1, \dots, N/2$ . Then Eqs. (8) can be written in the form (5) with  $N \rightarrow N/2$ ,  $L \rightarrow L/2$ , and the additional term  $\arctan(c/2k_i)$  on the right-hand side (see also Eq. (21) in [9]). As a result, the equations almost coincide with

(5). But we have no complete coincidence due to the term  $\arctan(c/2k_i)$ . Therefore, the solutions for zero BCs do not belong to the solution set for periodic BCs. Hence, the uniqueness of solutions with zero BCs should be proved separately. This problem is more complicated than that for periodic BCs.

As was noted above, it is sufficient to solve Eqs. (5) in the case where all  $k_j$  are different and positive. Therefore, we may always order  $k$ 's so that

$$0 < k_1 < k_2 < \dots < k_N. \quad (9)$$

It follows from (5) that

$$\begin{aligned} L(k_{i+1} - k_i) &= \pi(n_{i+1} - n_i) + 2 \arctan \frac{c}{k_{i+1} - k_i} \\ &+ \sum_{j=1}^{i-1} \left( \arctan \frac{c}{k_{i+1} - k_j} - \arctan \frac{c}{k_i - k_j} \right) + \sum_{j=1}^{i-1} \left( \arctan \frac{c}{k_{i+1} + k_j} - \arctan \frac{c}{k_i + k_j} \right) \\ &+ \sum_{j=i+2}^N \left( \arctan \frac{c}{k_{i+1} - k_j} - \arctan \frac{c}{k_i - k_j} \right) + \sum_{j=i+2}^N \left( \arctan \frac{c}{k_{i+1} + k_j} - \arctan \frac{c}{k_i + k_j} \right). \end{aligned} \quad (10)$$

At the ordering (9), each of four sums on the right-hand side of (10) is negative. Therefore, in order that  $k_{i+1} - k_i > 0$ , it is necessary that  $\pi(n_{i+1} - n_i) + 2 \arctan \frac{c}{k_{i+1} - k_i} > 0$ . From whence, we have  $n_{i+1} - n_i \geq 0$ . That is, the ordering (9) is possible only if

$$n_1 \leq n_2 \leq \dots \leq n_N. \quad (11)$$

Consider firstly the case where all  $n_i$  are different:

$$n_1 < n_2 < \dots < n_N. \quad (12)$$

Using the relation

$$\arctan \alpha = (\pi/2) \operatorname{sgn}(\alpha) - \arctan(1/\alpha) \quad (13)$$

and ordering (9), we now pass from (5) to the equivalent equations

$$Lk_i = \pi I_i - \sum_{j=1}^N \left( \arctan \frac{k_i - k_j}{c} + \arctan \frac{k_i + k_j}{c} \right) \Big|_{j \neq i}, \quad (14)$$

$$I_i = n_i + i - 1. \quad (15)$$

Since Eq. (13) is not defined for  $\alpha = 0$ , systems (14) and (5) are not equivalent for such collections  $\{k_i\}$ , for which  $k_i + k_j = 0$  or  $k_i - k_j = 0$  for some  $i \neq j$ . But this is insignificant. Since system (5) is not defined for  $k_i \pm k_j = 0$ , and system (14) has no solutions for  $k_i \pm k_j = 0$ . Indeed, relation  $k_i + k_j = 0$  requires  $I_i + I_j = 0$ , which means  $n_i + i + n_j + j = 2$ . But this is impossible in view of  $i, j, n_i, n_j \geq 1$ . The condition  $k_i - k_j = 0$  requires  $I_i - I_j = 0$ , i.e.,  $n_i + i = n_j + j$ . This is also impossible, since, for  $i < j$ , ordering (12) requires  $n_i < n_j$ , and, for  $i > j$ , the inequality  $n_i > n_j$  should hold. Therefore, we

may assert that (i) systems (14) and (5) are equivalent in the whole domain of existence of solutions, and (ii) their solutions coincide.

Following the Yang and Yang's idea [4], we introduce the function

$$B\{k\} = \sum_{j=1}^N \left( \frac{Lk_j^2}{2} - \pi I_j k_j \right) + \frac{1}{2} \sum_{j,l=1}^{N'} \left( \int_0^{k_j-k_l} \arctan(k/c) dk + \int_0^{k_j+k_l} \arctan(k/c) dk \right), \quad (16)$$

where the prime over the sum means  $j \neq l$ . On the whole interval of integration, the integrands are continuous. Hence, the integral can be differentiated with respect to the upper limit. The points of extremum of the function  $B(k_1, k_2, \dots, k_N)$  are set by the equations

$$\frac{\partial B}{\partial k_i} = 0, \quad i = 1, \dots, N, \quad (17)$$

which coincide with (14). The matrix of the second derivatives reads

$$B_{ij} = \frac{\partial^2 B}{\partial k_i \partial k_j} = \delta_{ij} \left( L - \frac{2c}{c^2 + 4k_i^2} + \sum_{l=1}^N \frac{c}{c^2 + (k_i - k_l)^2} + \sum_{l=1}^N \frac{c}{c^2 + (k_i + k_l)^2} \right) + \frac{c}{c^2 + (k_i + k_j)^2} - \frac{c}{c^2 + (k_i - k_j)^2} = B_{ji}, \quad (18)$$

where  $i, j = 1, \dots, N$ . This matrix is positive definite, since, for a set of any real numbers  $u_j$  and  $k_j$ , the following relation holds at  $c > 0$ :

$$\sum_{i,j=1}^N u_i B_{ij} u_j = \sum_{j=1}^N u_j^2 L + \sum_{j,l}^{j < l} \left[ \frac{c(u_j - u_l)^2}{c^2 + (k_j - k_l)^2} + \frac{c(u_j + u_l)^2}{c^2 + (k_j + k_l)^2} \right] \geq 0. \quad (19)$$

Moreover, the right-hand side of (19) is zero only if  $u_j = 0$  for all  $j$ .

Or otherwise, the necessary and sufficient condition of positive definiteness of a real symmetric matrix  $B_{ij}$  (with  $i, j = 1, \dots, N$ ) is as follows [11]:

$$G_j > 0, \quad j = 1, \dots, N, \quad (20)$$

where  $G_j$  are the dominant minors:

$$G_j = \det(B_{il}), \quad i, l = 1, \dots, j. \quad (21)$$

In particular,  $G_1 = B_{11}$ ,  $G_2 = B_{11}B_{22} - B_{12}B_{21}$ , and so on,  $G_N$  is the determinant of the entire matrix  $B_{ij}$ . We determined numerically minors  $G_j$  for various sets of  $\{k_i\}$  for  $N = 10, 100$ , and 1000. We have studied the homogeneous distributions of  $\{k_i\}$  and the inhomogeneous ones, with a small step  $\{\Delta k_i\}$  at the transition to the following configuration. For all sets of  $\{k_i\}$ , we have found

$$0 < G_1 < G_2 < \dots < G_N. \quad (22)$$

This ensure the validity of criterion (20).

Let us now forget condition (9) and consider that  $k_j \in [-\infty, \infty]$  for all  $j$ . In this case, the matrix of the second derivatives of the function  $B(k_1, k_2, \dots, k_N)$  (16) is positive definite, and we have  $B \rightarrow +\infty$  as  $k_i \rightarrow \pm\infty$  for all  $i$ . Such function  $B(k_1, k_2, \dots, k_N)$  must have only one stationary point (i.e., the point, at which relation (17) is satisfied), namely, a minimum. The second stationary point cannot exist: A maximum and a saddle point are excluded, because the matrix  $B_{ij}$  must not be positive definite in vicinities of these points. But, according to (19),  $B_{ij}$  is positive definite for any real  $k$ 's. The second minimum is impossible too, since at least one stationary point would exist between two minima; and, in a vicinity of this point, the matrix  $B_{ij}$  must not be positive definite. We conclude that the function  $B(k_1, \dots, k_N)$  has one and only one stationary point, and it is a minimum. Therefore, the system of equations (17) (and, hence, systems (14) and (5)) has one and only one real solution  $\{k_i\}$ .

Such conclusion is valid under condition (12), according to which all  $n_j$  are different. Let the set  $\{n_j\}$  has several identical  $n_j$ . In this case, the equations in system (5) corresponding to identical  $n_j$  are indistinguishable. This means that the numbers  $k_j$  corresponding to identical numbers  $n_j$  can be interchanged in the solution  $\{k_i\}$ . All such sets  $\{k_i\}$  are solutions of system (5). The ordering (9) is proper only for one set  $\{k_i\}$ . For the remaining sets, Eq. (5) leads to Eq. (14) with other sets of numbers  $I_i$ . However, all these solutions differ from one another only by a permutation of quasimomenta  $k_j$  and, therefore, are physically equivalent. We consider them as one solution.

Thus, we have proved that, for any set of real numbers  $n_i$ , the system of Gaudin's equations (5) has one and only one real solution  $\{k_i\}$ . Our analysis does not prove that the solutions for certain  $n$ 's satisfy condition (9). But the direct numerical solution of system (5) shows that, if the inequality  $0 < n_1 \leq n_2 \leq \dots \leq n_N$  holds, the solutions satisfy condition (9).

If we would construct  $B(k_1, \dots, k_N)$  on the basis of Eq. (5) instead of (14), then the first derivatives of  $B(k_1, \dots, k_N)$  would vary by a jump on the surfaces  $k_i \pm k_j = 0$ . These surfaces divide the manifold  $\{k_i\}$  into a huge number of domains. In this case, the matrix  $B_{ij}$  inside each domain is defined by formula (18) and is positive definite. But, due to a jump of the functions  $\partial B / \partial k_j$  on the boundaries of domains, the function  $B(k_1, \dots, k_N)$  can have *many* minima (at most one in each domain). In such approach, it is difficult to prove the uniqueness of solutions in the whole domain  $k_j \in [-\infty, \infty]$  (for all  $j$ ). The transition from system (5) to system (14) allows us to avoid this difficulty.

For Section 2, we need to prove also the uniqueness of the solution of system (7). The proof can be carried on like that for system (5). For the collection  $0 < k_2 < \dots < k_N$ , we pass from (7) to the equations

$$Lk_i = \pi I_i - 2 \arctan \frac{k_i}{c} - \sum_{j=2}^N \left( \arctan \frac{k_i - k_j}{c} + \arctan \frac{k_i + k_j}{c} \right) \Big|_{j \neq i} \quad (23)$$

with  $i = 2, \dots, N$  and  $I_i = n_i + i - 1$ . For system (23), we find

$$B\{k\} = \sum_{j=2}^N \left( \frac{Lk_j^2}{2} - \pi I_j k_j + 2 \int_0^{k_j} \arctan \frac{k}{c} dk \right) + \frac{1}{2} \sum_{j,l=2}^{N'} \left( \int_0^{k_j-k_l} \arctan \frac{k}{c} dk + \int_0^{k_j+k_l} \arctan \frac{k}{c} dk \right), \quad (24)$$

$$\begin{aligned} B_{ij} &= \delta_{ij} \left( L - \frac{2c}{c^2 + 4k_i^2} + \frac{2c}{c^2 + k_i^2} + \sum_{l=2}^N \frac{c}{c^2 + (k_i - k_l)^2} + \sum_{l=2}^N \frac{c}{c^2 + (k_i + k_l)^2} \right) + \\ &+ \frac{c}{c^2 + (k_i + k_j)^2} - \frac{c}{c^2 + (k_i - k_j)^2} = B_{ji}, \quad i, j = 2, \dots, N, \end{aligned} \quad (25)$$

$$\sum_{i,j=2}^N u_i B_{ij} u_j = \sum_{j=2}^N u_j^2 \left( L + \frac{2c}{k_j^2 + c^2} \right) + \sum_{j,l}^{j < l} \left[ \frac{c(u_j - u_l)^2}{c^2 + (k_j - k_l)^2} + \frac{c(u_j + u_l)^2}{c^2 + (k_j + k_l)^2} \right] \geq 0, \quad (26)$$

where  $\{k\} = k_2, k_3, \dots, k_N$ , and the prime above the sum in (24) means  $j \neq l$ . Similarly to the above analysis, we extend the admissible domain for each  $k_j$  to  $[-\infty, \infty]$ . Then, in view of formulae (23)–(26), we conclude that system (7) has the unique real solution  $(k_2, \dots, k_N)$ . It is possible to verify by direct numerical solution of Eqs. (7) that, provided  $0 < n_2 < n_3 < \dots < n_N$ , the inequalities  $0 < k_2 < k_3 < \dots < k_N$  are valid. If several  $n_i$  in (7) coincide, we have several physically equivalent solutions, which differ only by a permutation of quasimomenta  $k_i$ . As above, we consider them as one solution.

## 4 Conclusion

We have shown that, in order to get all solutions of Gaudin's equations (5), it is sufficient to consider the quantum numbers  $n_j = 1, 2, 3, \dots$  for all  $j$ . We have also proved that, for any set of different real numbers  $n_j$ , the system of equations (5) has the unique real solution  $\{k_j\}$ . If the numbers  $n_j$  include  $p$  identical ones ( $n_{l+1} = \dots = n_{l+p}$ ), then the Gaudin's system (5) has  $p!$  physically equivalent solutions  $\{k_j\}$ , which differ only by a permutation of quasimomenta  $k_{l+1}, \dots, k_{l+p}$ .

## Acknowledgments

The author thanks Yu. Besspalov for the discussion.

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